IX. MRA AND CONSTRUCTION OF WAVELETS (PART TWO)

Now we are ready to characterize the spaces V_0 and V_{-1} in a Multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$ using its scaling function and low pass filter.

Proposition 1. *Suppose that $\{V_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Multiresolution Analysis with φ being its scaling function and m_0 being the low pass filter induced by φ. Then*

$$V_0 = \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\},$$

$$V_{-1} = \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = m(2\xi)\hat{\varphi}(2\xi), m \in L^2([-\pi,\pi])\},$$

$$= \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = m(2\xi)m_0(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\}.$$

Proof. We first show that $V_0 \subset \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\}$. Since φ is a scaling function for the Multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$, namely $\{\varphi(x-l) | l \in \mathbb{Z}\}$ is a complete orthonormal system for V_0, so for any $f \in V_0$, there is a sequence of complex numbers $\{b_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$f(x) = \sum_{l\in\mathbb{Z}} b_l \varphi(x+l)$$

where the convergence is under the norm of $L^2(\mathbb{R})$. Hence by applying Lemma 4 of last chapter (together with Lemma 1 of last chapter), we have $\hat{f}(\xi) = \sum_{l\in\mathbb{Z}} b_l\hat{\varphi}(\xi)e^{il\xi}$ where the convergence is under the norm of $L^2(\mathbb{R})$. The fact that $\{b_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ implies that $\sum_{l\in\mathbb{Z}} b_l e^{il\xi} \in L^2([-\pi,\pi])$. Also, by Theorem 1 of last chapter, the fact that $\{\varphi(x-l) | l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$ implies that $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 = 1$ holds for any $\xi \in \mathbb{R}$. Thus, according to Theorem 2 of last chapter, we have

$$\hat{f}(\xi) = \sum_{l\in\mathbb{Z}} b_l\hat{\varphi}(\xi)e^{il\xi} = \hat{\varphi}(\xi)(\sum_{l\in\mathbb{Z}} b_l e^{il\xi}).$$

To prove that $\{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\} \subset V_0$, suppose for any fixed $f \in Ł^2(\mathbb{R})$, there is a function $m(\xi) \in L^2([-\pi,\pi])$, such that $\hat{f}(\xi) = \hat{\varphi}(\xi)m(\xi)$, then there is a sequence of complex numbers $\{b_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that $m(\xi) = \sum_{l\in\mathbb{Z}} b_l(\xi)e^{il\xi}$. Again, the fact that $\{\varphi(x-l) | l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, together with Theorem 1 and Theorem 2 of last chapter, implies,

$$\hat{f}(\xi) = \hat{\varphi}(\xi)(\sum_{l\in\mathbb{Z}} b_l e^{il\xi}) = \sum_{l\in\mathbb{Z}} b_l\hat{\varphi}(\xi)e^{il\xi}.$$

Finally, applying Lemma 4 to the equation $\hat{f}(\xi) = \sum_{l\in\mathbb{Z}} b_l \hat{\varphi}(\xi) e^{il\xi}$, we get that

$$f(x) = \sum_{l\in\mathbb{Z}} b_l \varphi(x+l)$$

where the convergence is under the norm of $L^2(\mathbb{R})$. This means that $f \in V_0$.

As for the characterization of subspace V_{-1}, we note that by using the definition of Multiresolution Analysis, Lemma 1 of last chapter and characterization of V_0, simple substitution of variables, and the fact that $\hat{\phi}(2\xi) = \hat{\phi}(\xi) m_0(\xi)$, in that order, we get that for any $f \in L^2(\mathbb{R})$,

$$\begin{aligned} f(x) \in V_{-1} &\Longleftrightarrow 2f(2x) \in V_0 \\ &\Longleftrightarrow \hat{f}(\frac{\xi}{2}) = m(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi]) \\ &\Longleftrightarrow \hat{f}(\xi) = m(2\xi)\hat{\varphi}(2\xi), m \in L^2([-\pi,\pi]) \\ &\Longleftrightarrow \hat{f}(\xi) = m(2\xi)m_0(\xi)\hat{\varphi}(\xi), m \in L^2([-\pi,\pi]). \end{aligned}$$

Hence both characterizations for V_{-1} are valid. $\square$

Next we find a characterization for W_{-1} with the help of the characterizations we just obtained for V_0 and V_{-1}.

Proposition 2. *Suppose that $\{V_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Multiresolution Analysis with φ being its scaling function and m_0 being the low pass filter induced by φ. Then*

$$W_{-1} = \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = e^{i\xi} m(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\}.$$

Proof. First we are to prove that

$$W_{-1} \subset \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = e^{i\xi} m(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\}.$$

To this end, let $f \in L^2(\mathbb{R})$ be a function in $W_{-1} = V_0 \ominus V_{-1}$, then $f \in V_0, f \perp V_{-1}$. According to Proposition 1, there exists a function $m_1 \in L^2([-\pi,\pi])$ such that $\hat{f}(\xi) = \hat{\varphi}(\xi)m_1(\xi)$. Moreover, for any $m \in L^2([-\pi,\pi])$, $\hat{f}(\xi)$ and $m(2\xi)m_0(\xi)\hat{\varphi}(\xi)$ are orthogonal to each other. Namely,

$$\int_{-\infty}^{\infty} \hat{\varphi}(\xi)m_1(\xi)\overline{m(2\xi)m_0(\xi)\hat{\varphi}(\xi)}d\xi = 0.$$

Using the technique of "periodization of integral" which we employed in the proof of Theorem1 and Theorem 2 of last chapter, we get that

$$\int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 m_1(\xi)\overline{m(2\xi)m_0(\xi)}d\xi = 0.$$

The detail is left for the reader. Since $\varphi(x)$ is a scaling function for $\{V_n\}_{n\in\mathbb{Z}}$ so by theorem 1 of last chapter, $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 = 1$. Hence

$$\int_{-\pi}^{\pi} m_1(\xi)\overline{m(2\xi)m_0(\xi)}d\xi = 0.$$

Thus

$$\int_{-\pi}^{0} m_1(\xi)\overline{m(2\xi)m_0(\xi)}d\xi + \int_{0}^{\pi} m_1(\xi)\overline{m(2\xi)m_0(\xi)}d\xi = 0.$$

With a change of variable, we have

$$\int_{0}^{\pi} \overline{m(2\xi)}[m_1(\xi)\overline{m_0(\xi)} + m_1(\xi-\pi)\overline{m_0(\xi-\pi)}]d\xi.$$

Note that $\overline{m(2\xi)}$ is an arbitrary π-periodical function with $\int_0^\pi |m(2\xi)|^2 d\xi < \infty$. Denote $g(\xi) = m_1(\xi)\overline{m_0(\xi)} + m_1(\xi-\pi)\overline{m_0(\xi-\pi)}$, then $g(\xi)$ is some specific π-periodical function with $\int_0^\pi |g(\xi)|^2 d\xi < \infty$ (prove it!). If we choose a m such at $m(2\xi) = g(\xi) = m_1(\xi)\overline{m_0(\xi)} + m_1(\xi-\pi)\overline{m_0(\xi-\pi)}$ then we get that

$$\int_{-\pi}^{\pi} |g(\xi)|^2 d\xi = 0$$

which means $g(\xi) = 0$ for all $\xi \in \mathbb{R}$. Namely for any ξ in $\mathbb{R}$,

$$m_1(\xi)\overline{m_0(\xi)} + m_1(\xi-\pi)\overline{m_0(\xi-\pi)} = 0.$$

This means that for each fixed $\xi \in \mathbb{R}$, vectors $(m_1(\xi), m_1(\xi-\pi))$ and $(\overline{m_0(\xi)}, \overline{m_0(\xi-\pi)}) \in \mathbb{C}^2$ are orthogonal to each other. Thus there is a function $\lambda : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$(m_1(\xi), m_1(\xi-\pi)) = \lambda(\xi)(\overline{m_0(\xi-\pi)}, -\overline{m_0(\xi)})$$

Since $m_1(\xi)$ and $m_0(\xi)$ are both $2\pi-$periodical functions, and for each $\xi \in \mathbb{R}$, the norm of the vector $(\overline{m_0(\xi-\pi)}, -\overline{m_0(\xi)})$ is always 1 (see Theorem 3 of last chapter), so $\lambda(\xi)$ is a $2\pi-$periodical function. Note also that for each $\xi \in \mathbb{R}$,

$$(m_1(\xi-\pi), m_1(\xi)) = \lambda(\xi-\pi)(\overline{m_0(\xi)}, -\overline{m_0(\xi-\pi)}),$$

so we see that$\lambda(\xi) = -\lambda(\xi-\pi)$ for any $\xi \in \mathbb{R}$. Hence $e^{-i\xi}\lambda(\xi)$ is a $\pi-$periodical function. It follows that $l(\xi) = e^{-i\frac{\xi}{2}}\lambda(\frac{\xi}{2})$ is a 2π-periodical function. Lastly, since $m_1(\xi) = \lambda(\xi)\overline{m_0(\xi-\pi)}$, we can compute (The detail is left for the reader) to get that

$$\int_{-\pi}^{\pi} |m_1(\xi)|^2 d\xi = 2\int_{0}^{\pi} |\lambda(\xi)|^2 d\xi = \int_{\pi}^{\pi} |\lambda(\xi)|^2 d\xi.$$

So $\lambda(\xi) \in L^2([-\pi,\pi])$. Whence

$$\int_{-\pi}^{\pi} |l(\xi)|^2 d\xi = \int_{-\pi}^{\pi} |\lambda(\frac{\xi}{2})|^2 d\xi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\lambda(u)|^2 2du = \int_{-\pi}^{\pi} |\lambda(\xi)|^2 d\xi < \infty,$$

therefore $l(\xi) \in L^2([-\pi,\pi])$. Thus we have

$$\hat{f}(\xi) = m_1(\xi)\hat{\varphi}(\xi) = \lambda(\xi)\overline{m_0(\xi-\pi)}\hat{\varphi}(\xi) = e^{i\xi} l(2\xi)\overline{m_0(\xi-\pi)}\hat{\varphi}(\xi).$$

To prove that $\{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = e^{i\xi} m(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi), m \in L^2([-\pi,\pi])\} \subset W_{-1}$, suppose that $f \in L^2(\mathbb{R})$ is such a function such that $\hat{f}(\xi) = e^{i\xi} m(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi)$ with some $m(\xi) \in L^2([-\pi,\pi])$. With the help of Proposition 1, we will be able to prove that $f \in V_0$ and $f \perp V_{-1}$ hence $f \in W_{-1}$. The details are left for the reader. $\square$

Now a characterization for the space W_0 is immediate.

Theorem 1. *Suppose that $\{V_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Multiresolution Analysis with φ being its scaling function and m_0 being the low pass filter induced by φ. Then*

$$W_0 = \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = e^{i\frac{\xi}{2}} m(\xi)\overline{m_0(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2}), m \in L^2([-\pi,\pi])\}.$$

Proof. Note that by Lemma 1 of chapter 7, we have that for any $n \in \mathbb{Z}$,

$$f(x) \in W_n \Longleftrightarrow f(2x) \in W_{n+1}.$$

Now a characterization of W_0 can be derived from that of W_{-1} the same way as in the proof of Proposition 1. Details are left for the reader. $\square$

The next theorem shows that using the characterization above for W_0, it is very easy to identify such function $\psi \in W_0$ that $\{\psi(x+l) | l \in \mathbb{Z}\}$ forms a complete orthonormal system for W_0.

Theorem 2. *Suppose that $\{V_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Multiresolution Analysis with φ being its scaling function and m_0 being the low pass filter induced by φ. Let $\psi \in W_0$ be an arbitrary function. Then the following two statements are equivalent:*

a)$\{\psi(x+l) | l \in \mathbb{Z}\}$ forms a complete orthonormal system for W_0;

b)$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\gamma(\xi)\overline{m_0(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2})$ for some function $\gamma \in L^2([-\pi,\pi])$ such that $|\gamma(\xi)| \equiv 1$ for any $\xi \in \mathbb{R}$.

Proof. First of all, according to the characterization of W_0 above, for any function $\psi \in W_0$, there is a $\gamma \in L^2([-\pi,\pi])$ such that

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\gamma(\xi)\overline{m_0(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2}).$$

To prove the implication a)$\Longrightarrow$ *b*), we only need to show that if $\{\psi(x+l)|l \in \mathbb{Z}\}$ is a orthonormal system in $L^2(\mathbb{R})$, then function γ above satisfies $|\gamma(\xi)| \equiv 1$ for any $\xi \in \mathbb{R}$. Note that by Theorem 1 of chapter 8, we have that $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 \equiv 1$ for any $\xi \in \mathbb{R}$. This property is also enjoyed by φ since $\varphi(x)$ is a scaling function. Using this property, together with that of low pass filter $m_0(\xi)$ stated in Theorem 3 of chapter 8, we do the computation as follows

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |\gamma(\xi)|^2 \cdot |m_0(\frac{\xi}{2}+k\pi+\pi)|^2 \cdot |\hat{\phi}(\frac{\xi}{2}+k\pi)|^2$$

$$= |\gamma(\xi)|^2 (\sum_{k\in\mathbb{Z}}^{k=2l} |m_0(\frac{\xi}{2}+k\pi+\pi)|^2 |\hat{\phi}(\frac{\xi}{2}+k\pi)|^2 + \sum_{k\in\mathbb{Z}}^{k=2l+1} |m_0(\frac{\xi}{2}+k\pi+\pi)|^2 |\hat{\phi}(\frac{\xi}{2}+k\pi)|^2)$$

$$= |\gamma(\xi)|^2 (\sum_{l\in\mathbb{Z}} |m_0(\frac{\xi}{2}+2l\pi+\pi)|^2 |\hat{\phi}(\frac{\xi}{2}+2l\pi)|^2 + \sum_{l\in\mathbb{Z}} |m_0(\frac{\xi}{2}+2l\pi+2\pi)|^2 |\hat{\phi}(\frac{\xi}{2}+2l\pi+\pi)|^2)$$

$$= |\gamma(\xi)|^2 (\sum_{l\in\mathbb{Z}} |m_0(\frac{\xi}{2}+\pi)|^2 |\hat{\phi}(\frac{\xi}{2}+2l\pi)|^2 + \sum_{l\in\mathbb{Z}} |m_0(\frac{\xi}{2})|^2 |\hat{\phi}(\frac{\xi}{2}+2l\pi+\pi)|^2)$$

$$= |\gamma(\xi)|^2 (|m_0(\frac{\xi}{2}+\pi)|^2 + |m_0(\frac{\xi}{2})|^2) = |\gamma(\xi)|^2.$$

As for the implication b)$\Longrightarrow$ *a*), if we assume $|\gamma(\xi)|^2 = 1$ for all $\xi \in \mathbb{R}$, with $\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\gamma(\xi)\overline{m_0(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2})$, we can similarly compute to get that $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 \equiv 1$. Hence by Theorem 1 of chapter 8, $\{\psi(x+l)|l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. Furthermore, for any function $g \in W_0$, according to the characterization of W_0, there is a function $s \in L^2([-\pi,\pi])$ such that

$$\hat{g}(\xi) = e^{i\frac{\xi}{2}} s(\xi)\overline{m_0(\frac{\xi}{2}+\pi)}\hat{\varphi}(\frac{\xi}{2}).$$

Since $|\gamma(\xi)|^2 = 1$, so

$$\hat{g}(\xi) = s(\xi)\overline{\gamma(\xi)}\hat{\psi}(\xi).$$

Clearly $s(\xi)\overline{\gamma(\xi)} \in L^2([-\pi,\pi])$. So there is a sequence of complex numbers $\{c_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$s(\xi)\overline{\gamma(\xi)} = \sum_{l\in\mathbb{Z}} c_l e^{il\xi}.$$

Thus, by Theorem 2 of chapter 8, we have

$$\hat{g}(\xi) = (\sum_{l\in\mathbb{Z}} c_l e^{il\xi})\hat{\psi}(\xi) = \sum_{l\in\mathbb{Z}} c_l \hat{\psi}(\xi) e^{il\xi}.$$

Whence by Lemma 4 (together with Lemma 1) of chapter 8, we have that

$$g(x) = \sum_{l\in\mathbb{Z}} c_l \psi(x+l)$$

where the convergence is under the norm of $L^2(\mathbb{R})$. This means that $\{\psi(x+l) | l \in \mathbb{Z}\}$ is a complete orthonormal system in W_0. $\square$

Lastly, let us talk about the concrete way of constructing ψ using φ and m_0. Recall the way how m_0 is induced by φ, we let $\{\alpha_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be the sequence of complex numbers such that

$$\frac{1}{2}\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} \alpha_k \varphi(x+k),$$

then $\hat{\varphi}(2\xi) = \hat{\varphi}(\xi) m_0(\xi)$ where $m_0(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\xi}$ therefore

$$\overline{m_0(\xi+\pi)} = \sum_{k\in\mathbb{Z}} \overline{\alpha}_k e^{-ik\xi}(-1)^k.$$

In Theorem 2 above, if we take $\gamma(\xi) \equiv 1$, then a function ψ is a wavelet if

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi) = \sum_{k\in\mathbb{Z}} \overline{\alpha}_k e^{-i(k-1)\xi}(-1)^k \hat{\varphi}(\xi).$$

Hence

$$\frac{1}{2}\psi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} \overline{\alpha}_k (-1)^k \varphi(x-(k-1))$$

or

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} \overline{\alpha}_k (-1)^k \varphi(2x-(k-1)).$$

Take Haar wavelet as an example, if we let $\varphi(x) = \chi_{[0,1)}(x)$, then

$$\frac{1}{2}\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} \alpha_k \varphi(x+k) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-1),$$

thus $\alpha_0 = \alpha_{-1} = \frac{1}{2}$ and $\alpha_k = 0$ whenever $k \neq 0$ and $k \neq -1$. Thus, by the formula above, we see that

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} \overline{\alpha}_k (-1)^k \varphi(2x-(k-1))$$

$$= 2\cdot\frac{1}{2}\varphi(2x+1) + 2\cdot\frac{1}{2}\cdot(-1)\varphi(2x+2)$$

$$= \chi_{[-\frac{1}{2},0)} - \chi_{[-1,-\frac{1}{2})}.$$